Enrollment No: ______________________ **Exam Seat No:**______________________

C. U. SHAH UNIVERSITY

Summer Examination-2022

Subject Name : Linear Algebra

Subject Code : 5SC01LIA1 **Branch: M.Sc. (Mathematics)**

Semester: 1 **Date: 21/04/2022** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions.** **[07]**

a. Define: External Direct Sum. **(02)**
b. Define: Dual Space. **(02)**
c. Define:Minimal Polynomial of T. **(02)**
d. True/False : $W^{00} = W$. **(01)**

Q-2 **Attempt all questions** **[14]**

a. Let V and W be vector space over F of dimension m and n respectively. Then prove that $HOM(V,W)$ is of dimension mn over F. **(07)**
b. If $v_1, v_2, \dots, v_n \in V$ then prove that either they are linearly independent or some v_k is a linear combination of preceding one's $v_1, v_2, \dots \dots \dots . v_{k-1}$ **(04)**
c. If $v_1, v_2, \dots \dots \dots . v_n \in V$ are linearly independent then prove that every element in their linear span has a unique representation in the form $\lambda_1 v_1 + \lambda_2 v_2 + \cdots \dots \dots . +\lambda_n v_n$ with $\lambda_i \in F$. **(03)**

OR

Q-2 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over F and W be subspace of V. Show that $\widehat{W}$ is isomorphic to $\hat{V}/_{W^\circ}$ and **(09)**

$$\dim W^\circ = \dim V - \dim W.$$

b. If $\{v_1, v_2, \dots, v_n\}$ is a basis of V over F and if $w_1, w_2, \dots \dots \dots ., w_m$ are linearly independent over F then prove that $m \leq n$. **(05)**

Q-3 **Attempt all questions.** **[14]**

a. If $\mathcal{A}$ is an algebra over F with unit element then prove that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. **(06)**
b. Let$V$be a finite dimensional over F then prove that $T \in A(V)$ is invertible **(05)**

if and only if the constant term in minimal polynomial for T is nonzero.

c. Let V be finite dimensional over F and $T \in A(V)$.Show that the number of characteristic roots of T is atmost n^2. **(03)**

OR

Q-3 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over Fthen prove that $T \in A(V)$ is regular if and only if $T$is one-one. **(05)**

b. Let V be finite dimensional over F and $T \in A(V)$. If $\lambda_1, \lambda_2, \dots \dots \dots . \lambda_k$ in F are distinct roots of T and $v_1, v_2, \dots \dots \dots . v_k$ are characteristic vector of T corresponding to $\lambda_1, \lambda_2, \dots \dots \dots . \lambda_k$ respectively. Then prove that $v_1, v_2, \dots \dots \dots . v_k$are linearly independent. **(05)**

c. Let$V$be finite dimensional over F and $S, T \in A(V)$ and S be regular, then prove that $\lambda \in F$ is chatracteristic root of T if and only if it is a characteristic root of $S^{-1}TS$. **(04)**

SECTION – II

Q-4 **Attempt the Following questions.** **[07]**

a Prove or disprove :$tr(A + B) = tr(A) + tr(B) where A, B \in M_n(F)$. **(02)**

b Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$,find det A.State the result you use. **(02)**

c Find the symmetric matrix associated with the quadratic form **(02)**

$$9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + 2x_2x_3 .$$

d Define: Index of Nilpotence. **(01)**

Q-5 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F then there is a basis of V with respect to which the matrix of T is upper triangular. **(07)**

b. Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent with index of nilpotence n_1.Then show that there is a basis of V in which the matrix of T is of the form **(05)**

$$\begin{pmatrix} M_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{n_k} \end{pmatrix}$$

Where $n_1 \geq n_2 \geq \cdots \geq n_k$ and $\dim V = n_1 + n_2 + \cdots n_k$.

c. Let V be a finite dimensional vector space over F. If $T \in A(V)$ is right invertible then show that$T$ is invertible. **(02)**

OR

Q-5 **Attempt all questions** **[14]**

a. Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent then prove that the invariants of T are unique. **(06)**

b. Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Then show that $\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m$is invertible if $\alpha_0 \neq 0$,where $\alpha_0, \alpha_1, \dots, \alpha_m \in F$. **(05)**

c. Show that two similar matrices have same trace. **(03)**

Q-6 **Attempt all questions** **[14]**

	a.	Let $A, B \in M_n(F)$,prove that $\det(AB) = \det A \cdot \det B$	**(07)**
	b.	Prove that the determinant of an upper triangular matrix is the product of its entries on the main diagonal.	**(07)**
		OR	
Q-6		**Attempt all questions**	**[14]**
	a.	State and prove Cramer's rule.	**(07)**
	b.	Prove that interchanging the two row of matrix changes the sign of its determinant.	**(04)**
	c.	IfA is regular then show that $\det A \neq 0$.	**(03)**